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# Engineering Notes

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## Stability Behavior of Linear Time-Varying Systems

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### I. Introduction

THE stability behavior of linear systems has been studied by many researchers (see, e.g., Refs. 1–4). Most of the stability criteria are obtained by means of Lyapunov functions. Recently, Hsu and Wu[3] and Shrivastava and Pradeep[4] gave many stability criteria for second-order linear time-varying systems by either defining a Lyapunov function or a quadratic function. Since the second-order linear system is a special case of the first-order linear system, it is natural to ask if these stability criteria can be consolidated, improved, and then extended to the first-order linear system. This is the objective of this Note.

In this Note, one stability criterion and two instability criteria for the first-order linear system are derived. In contrast to the implicit Lyapunov function, these criteria give explicit inequalities related to the coefficients of the system. A series of corollaries are obtained from Theorems 1, 2, and 3 for second-order linear systems. These results are improvements of the corresponding theorems in Refs. 3 and 4.

This Note is comprised of six sections. The definitions, notations, and some properties of a quadratic function used here are given in Sec. II. In Sec. III, most of stability criteria for second-order linear systems in Refs. 3 and 4 are consolidated, improved, and then extended to the first-order linear system as stated in Theorem 1. Section IV gives two instability criteria for first-order linear systems, followed by a series of instability criteria for second-order linear systems. Section V discusses an example of a new stability criterion for the linearized equation of motion of pendulum. A brief conclusion is given in the last section.

### II. Mathematical Notations and Definitions

Throughout this Note, the inner product of two vectors $X$ and $Y$ in $R^n$ is defined and denoted by

$$(X,Y) = X^TY = x_1y_1 + x_2y_2 + \dots + x_ny_n$$

where $X^T = (x_1, x_2, \dots, x_n)$ and $Y^T = (y_1, y_2, \dots, y_n)$ are the transpose of $X$ and $Y$, respectively. A matrix $P$ in $R^{n\times n}$ is called positive (negative) definite if $(x, Px) > 0$ $(< 0)$ for all $x \in R^n$. A matrix $P$ in $R^{n\times n}$ is called semipositive (seminegative) if $(x, Px) \geq 0$ $(\leq 0)$ for all $x \in R^n$. The norm of a vector $X$ in $R^n$ and a matrix $P$ in $R^{n\times n}$ is defined and denoted as

$$\|P\| = \max_{\|X\|=1} \|AX\|, \qquad \|X\| = (X,X)$$

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To simplify the notation, the explicit time dependence of the elements of these matrices or vectors is to be understood unless stated otherwise. The derivatives of a vector $X$ are denoted by $X' = \mathrm{d}X/\mathrm{d}t$ and $X'' = \mathrm{d}^2X/(\mathrm{d}t^2)$.

Consider the first-order linear system

$$X' = AX \tag{1}$$

where $A \in R^{n\times n}$ and the second-order linear system

$$MZ'' + QZ' + KZ = 0 \tag{2}$$

where $M$, $K$, and $Q$ are in $R^{n\times n}$. It is well known that Eq. (2) with nonsingular matrix $M$ can be reduced to Eq. (1) by choosing

$$X = \begin{pmatrix} Z \\ Z' \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}Q \end{pmatrix}$$

System (1) is said to be stable if all solutions of (1) are bounded as $t \to \infty$. On the other hand, if all solutions of (1) are unbounded, then system (1) is called unstable.

In this Note, a quadratic function $v = (X,PX)$ is constructed that has the following properties:

1) If $v' \leq 0$, $v > 0$, and $P \geq \epsilon I$ for all $t > t_0$, then $\|X\|$ is bounded.

2) If $v' > \epsilon$ and $0 < P \leq \epsilon I$ for all $t > t_0$ and for some positive constants $\epsilon$ and $t_0$, then $\|X\|$ is unbounded.

In property 1, if $P \geq \epsilon I$ is replaced by $P > 0$, then $\|X\|$ is not necessarily bounded. For example, $p = t^{-3}$, $v = t^{-1}$, and $X = t$.

In property 2, if $v' > \epsilon$ is replaced by $v' > 0$, then $v$ is not necessarily unbounded. For example, $v = \exp(-t^{-1})$. Also, without the condition on $P$ $(0 < P \leq \epsilon I)$, one can not guarantee the instability of $\|X\|$. For example: $p = t^3$, $v = t$, and $x = t^{-1}$.

We shall define the following conditions as "assumption A": Let $c$, $\epsilon$, and $t_0$ be positive constants and let $\alpha(t)$ be a scalar function. There exist time-varying matrices $M$ and $K$ in $R^{n\times n}$ such that $M^T = M \geq \epsilon I$, $K^T = K \geq \epsilon I$, and $\int_0^t \alpha(s)\,\mathrm{d}s < c$, for all $t > t_0$.

### III. Stability Criteria

It is well known that stability criteria of System (1) can be obtained by means of Liapunov functions. By constructing a Liapunov function or a quadratic function, Hsu and Wu[3] and Shrivastava and Pradeep[4] obtained many stability criteria for system (2). These stability criteria can be consolidated, improved, and then extended to system (1) as stated later.

*Theorem 1.* Let $c$, $\epsilon$, and $t_0$ be positive constants, and let $\alpha(t)$ be a scalar function. Assume that there exists a matrix $P(t)$ in $R^{n\times n}$ such that for all $t > t_0$

$$P' + A^TP + PA \leq \alpha(t)P \tag{3}$$

$$P \geq \epsilon I \tag{4}$$

$$\int_0^t \alpha(s)\mathrm{d}s < c \tag{5}$$

Then system (1) is stable.

*Proof:* For any given matrix $P$ in $R^{n\times n}$, define a quadratic function $v = (X, PX)$, for $X$ in system (1). Differentiate $v$ and apply inequality (3) to obtain $v' = [X, (P' + A^TP + PA)X] \leq \alpha(t)v$.

Solving for $v$ and applying inequality (4), we obtain

$$\epsilon \| X(t) \| < v(t) < v(t_0) \exp \left[ \int_0^t \alpha(s)\, ds \right]$$

It follows system (1) is stable because of inequality (5).

*Remark 1.* Theorem 1 is asymptotically stable if the condition $\int_0^\infty \alpha(s)\, ds < c$ in assumption A is replaced by $\int_0^\infty \alpha(s)\, ds = -\infty$.

By Theorem 1 with $P = I$, we obtain the following result.

*Corollary 1.* Let $c$ and $t_0$ be positive constants. Assume that there exists a scalar function $\alpha(t)$ such that $A \leq \alpha(t)I$ and $\int^t \alpha(s)\, ds < c$, for all $t > t_0$. Then system (1) is stable.

*Remark 2.* If $\alpha(t) = 0$, then the earlier corollary can be simplified as follows: If $A \leq 0$, then system (1) is stable.

Applying Theorem 1 to system (2), we obtain the following corollaries, which are improvements of results in Refs. 3 and 4.

*Corollary 2.* Let $\epsilon_1$, $\epsilon$, and $t_0$ be positive constants, and let $\alpha(t)$ be a scalar function. Assume that there exists a matrix $S$ such that $S > \epsilon I$, $C > \epsilon I$, $SC = (SC)^T$, $\int_0{}^t \alpha(s)\, ds < \epsilon_1$, $(C^T SC)' \leq \alpha(t) C^T SC$, and $(SC)' - 2SCB \leq \alpha(t) SC$, for all $t > t_0$. Then system $q'' + Bq' + Cq = 0$ is stable.

*Proof*: The proof follows from Theorem 1 by choosing

$$P = \begin{pmatrix} C^T SC & 0 \\ 0 & SC \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & I \\ -C & -B \end{pmatrix}$$

*Corollary 3.* Assume that assumption A holds. Further, suppose $K' \leq \alpha(t)K$, and $M' - 2D \leq \alpha(t)M$ for all $t > t_0$. Then $Mq'' + Dq' + Kq = 0$ is stable.

*Proof*: The proof follows from Theorem 1 by choosing

$$P = \begin{pmatrix} K & 0 \\ 0 & M \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{pmatrix}$$

*Corollary 4.* Suppose assumption A holds. Further, suppose

$$\begin{pmatrix} M' & -A^T K^{-1} M \\ -MK^{-1}A & (MK^{-1}M)' - 2MK^{-1}(D + G) \end{pmatrix} \leq \alpha(t) \begin{pmatrix} M & 0 \\ 0 & MK^{-1}M \end{pmatrix}$$

for all $t > t_0$. Then $Mq'' + (D + G)q' + (K + A)q = 0$ is stable.

*Proof*: The proof follows from Theorem 1 by choosing

$$P = \begin{pmatrix} M & 0 \\ 0 & MK^{-1}M \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & I \\ -M^{-1}(K + A) & -M^{-1}(D + G) \end{pmatrix}$$

*Corollary 5.* Suppose assumption A holds. Further, suppose

$$\begin{pmatrix} K' & -A^T \\ -A & M' - (D + G)^T - (D + G) \end{pmatrix} \leq \alpha(t) \begin{pmatrix} K & 0 \\ 0 & M \end{pmatrix}$$

for all $t > t_0$. Then $Mq'' + (D + G)q' + (K + A)q = 0$ is stable.

*Proof*: The proof follows from Theorem 1 by choosing

$$P = \begin{pmatrix} K & 0 \\ 0 & M \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & I \\ -M^{-1}(K + A) & -M^{-1}(D + G) \end{pmatrix}$$

*Remark 3.* Corollaries 2–5 with $\alpha(t) = 0$ and some (skew) symmetry conditions on the coefficients are results of Theorems 2–5, respectively, in Ref. 4. Also, Theorem 1 in Ref. 4 is a special case of Corollary 4 with $A = 0$. Further, Theorems 2 and 3 in Ref. 3 can be obtained from Corollary 3 by choosing $\alpha(t) = 0$ and $\alpha(t) = \max\{\|M' - 2D\|, \|K'\|\}$.

## IV. Instability Criteria

Similar to Theorem 1, by constructing the quadratic function $v(t)$, we obtain two instability criteria for System (1).

*Theorem 2.* Let $c$, $\epsilon$, and $t_0$ be positive constants and let $\alpha(t)$ be a scalar function. Assume that there exists a positive matrix $P$ in $R$ such that $P \leq \epsilon I$, $P' + A^T P + PA \geq \alpha P$, and $\int_0^\infty \alpha(s) ds = \infty$ for all $t > t_0$. Then system (1) is unstable.

*Proof*: From the proof of Theorem 1, $v' = [X, (P' + A^T P + PA)X] \geq \alpha(t)v$. Solving for $v$ and then applying the hypothesis results in

$$\epsilon \| X(t) \| \geq v(t) \geq v(0) \exp \left( \int_0^t \alpha(s) ds \right)$$

This shows that $\|X(t)\|$ is unbounded as $t \to \infty$, which implies system (1) is unstable.

By choosing $P = I$, Theorem 2 can be simplified as follows:

*Corollary 6.* If $A = A^T$ and $\int_0^t \lambda_{\min}[A(s)] ds = \infty$, where $\lambda_{\min}(A)$ is the smallest eigenvalue of $A$, then System (1) is unstable.

*Corollary 7.* Let $\epsilon$ and $t_0$ be positive constants. Assume that there exists a scalar function $\alpha(t)$ in $R$ such that

$$K' \geq \alpha K, \qquad M' - 2D \geq \alpha M \qquad 0 < K^T = K < \epsilon I$$

$$0 < M^T = M < \epsilon I \qquad \int_0^t \alpha(s) ds = \infty$$

for all $t > t_0$. Then system $Mq'' + Dq' + Kq = 0$ is unstable.

*Proof*: The proof follows from Theorem 2 by choosing

$$P = \begin{pmatrix} K & 0 \\ 0 & M \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{pmatrix}$$

*Remark 4.* If $\alpha(t) = \min\{\lambda_{\min}(K'), \lambda_{\min}(M' - 2D)\}$, then Corollary 7 becomes Theorem 4 in Ref. 3.

*Corollary 8.* If $Q = Q^T$ and $\int_0^\infty \lambda_{\min}[Q(s)] ds = \infty$, then $X'' - QX = 0$ is unstable.

*Proof*: The proof follows from Theorem 2 by choosing

$$P = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & I \\ Q & 0 \end{pmatrix}$$

*Remark 5.* The condition $Q(t) \geq \epsilon > 0$ in Theorem 5 of Ref. 3 is replaced by a weaker condition in Corollary 8.

*Theorem 3.* Suppose that there exist positive constants $c$, $c_1$, and $t_0$, such that $0 < P < c_1 I$ and $P' + A^T P + PA > cI$, for all $t > t_0$. Then System (1) is unstable.

*Proof*: From the proof of Theorem 1 and $0 < P \leq c_1 I$,

$$v' = [X, (P' + A^T P + PA)X] \geq c\,(X, X) \geq c_2 v,$$
$$c_2 = c/c_1 > 0$$

Solving for $v$ results in $c_1 \| X(t)\| \geq v(t) \geq v(t_0) \exp[c_2(t - t_0)]$. Thus system (1) is stable because $\|X(t)\| \to \infty$ as $t \to \infty$.

*Remark 6.* If $P = I$, Theorem 3 is a special case of Theorem 2.

*Corollary 9.* Suppose that there exist positive constants $c$, $c_1$, and $t_0$, such that $K' > cI$, $M' - 2D > cI$, $0 < K^T = K < c_1$, $0 < M^T = M < c_1 I$, for all $t > t_0$. Then system (2) is unstable.

*Proof*: The proof follows from Theorem 3 by choosing $A$ and $P$ as in Corollary 7.

## V. Example

Consider the following linearized equation of motion of a pendulum with time-varying length $a(t)$ as discussed in Refs. 3 and 4:

$$y'' + 2(a/a)y' + (g/a)y = 0 \tag{6}$$

where $g$ is the gravitational constant.

*Claim*: Assume that $a(t)$ is a positive bounded above function. If $a(t)$ satisfies one of the following conditions for all $t > t_0$

1) $a' \geq 0$
2) $a' \leq 0$
3) $a' + t^{-\beta}a \geq 0$, for some $\beta > 1$

then (6) is stable.

*Proof*: Criteria 1 and 2 can be proved by Corollary 3 with $M = 1, D = 2a'/a, K = g/a$, and $\alpha(t) = \max\{-a'/a, -4a'/a\}$. If $a' \geq 0$, then $\alpha(t) = -a'/a$. If $a' \leq 0$, then $\alpha(t) = -4a'/a$. This shows that $\int^{\infty}\alpha(s)\,ds < c$, which implies (6) is stable.

Criterion 3 can be obtained by Corollary 4 with $A = 0$, $M = 1, D + G = 2a'/a, K = g/a$, and $\alpha(t) = t^{-\beta}, \beta > 1$. The claim is then proved.

*Remark 7*. Criteria 1 and 2 can also be seen in Hsu and Wu.[3]

## VI. Conclusion

One stability criterion and two instability criteria for the first-order linear time-varying system are given in this Note. These criteria are extensions and/or consolidations of the results in Refs. 3 and 4. These conditions, though not intuitive, can be checked easily for a given system.

A general necessary and sufficient condition on stability is very difficult to derive. However, it is interesting to know if such conditions can be obtained in some forms of linear systems.

## References

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# Maximum Entropy Controller Synthesis for Colocated and Noncolocated Systems

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## I. Introduction

MAXIMUM entropy controller synthesis was developed specifically for the robust control of flexible structures.[1–4] The goal of this paper is to provide well-documented numerical examples that illustrate the characteristics of the method. The examples we consider in this note were chosen to contrast the properties of maximum entropy controllers in two key cases, namely, colocation and noncolocation. Our results confirm previous observations, namely, that maximum entropy controllers employ positive real phase stabilization in the colocated case and wider and deeper notch gain stabilization in the noncolocated case. The computations were performed using a standard quasi-Newton technique in conjunction with the appropriate cost gradient expressions.

## II. Maximum Entropy Controller Synthesis

Consider the structural model

$$\dot{x} = \left(A + \sum_{i=1}^{r} \sigma_i A_i\right) + Bu + D_1 w \tag{1}$$

$$y = Cx + D_2 w \tag{2}$$

with feedback controller

$$\dot{x}_c = A_c x_c + B_c y \tag{3}$$

$$u = C_c x_c \tag{4}$$

performance variables

$$z = E_1 x + E_2 u \tag{5}$$

and performance measure

$$J(A_c, B_c, C_c) = \lim_{t\to\infty} \mathscr{E}\left\{\frac{1}{t}\int_0^t z^T(s)z(s)\,ds\right\} \tag{6}$$

where $x \in \mathscr{R}^n$, $u \in \mathscr{R}^m$, $y \in \mathscr{R}^\ell$, $w \in \mathscr{R}^d$, $z \in \mathscr{R}^q$, $x_c \in \mathscr{R}^{n_c}$, $\sigma_i$ is an uncertain parameter representing uncertainty in $\omega_{di}$, and

$$A_i = \text{diag}\left\{0, \ldots, 0, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 0, \ldots, 0\right\} \tag{7}$$

so that the $i$th $2 \times 2$ diagonal block is the only nonzero entry in $A_i$. The disturbance $w$ is a standard white noise signal and $\mathscr{E}$ denotes expectation. The matrix $A$ is assumed to be in real normal coordinates, that is,

$$A = \text{diag}\left\{\begin{bmatrix} -\eta_1 & \omega_{d1} \\ -\omega_{d1} & -\eta_1 \end{bmatrix}, \ldots, \begin{bmatrix} -\eta_r & \omega_{dr} \\ -\omega_{dr} & -\eta_r \end{bmatrix}\right\} \tag{8}$$

In maximum entropy theory, the performance $J(A_c, B_c, C_c)$ is given by

$$J(A_c, B_c, C_c) = \text{tr}\,\tilde{Q}\tilde{E}^T\tilde{E} \tag{9}$$

where $\tilde{Q}$ satisfies the maximum entropy covariance equation

$$0 = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \sum_{i=1}^{r} \delta_i^2\left[\frac{1}{2}\tilde{A}_i^2\tilde{Q} + \tilde{A}_i\tilde{Q}\tilde{A}_i^T + \frac{1}{2}\tilde{Q}\tilde{A}_i^{2T}\right] + \tilde{D}\tilde{D}^T \tag{10}$$

and where $\tilde{A}$, $\tilde{A}_i$, $\tilde{D}$, and $\tilde{E}$ are defined by

$$\tilde{A} \triangleq \begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}, \quad \tilde{A}_i \triangleq \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{D} \triangleq \begin{bmatrix} D_1 \\ B_cD_2 \end{bmatrix}$$
$$\tilde{E} \triangleq [E_1 E_2 C_c] \tag{11}$$

and $\delta_i$ is a measure of the magnitude of the uncertainty $\sigma_i$.

To minimize $J(A_c, B_c, C_c)$ given by Eq. (9) where $\tilde{Q}$ satisfies Eq. (10), we define a Lagrangian function

$$\mathscr{L}(A_c, B_c, C_c, \tilde{Q}) \triangleq \text{tr}\,\tilde{Q}\tilde{E}^T\tilde{E} + \text{tr}\,\tilde{P}\Bigg(\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \sum_{i=1}^{r} \delta_i^2\left[\frac{1}{2}\tilde{A}_i^2\tilde{Q} + \tilde{A}_i\tilde{Q}\tilde{A}_i^T + \frac{1}{2}\tilde{Q}\tilde{A}_i^{2T}\right] + \tilde{D}\tilde{D}^T\Bigg) \tag{12}$$

where $\tilde{P}$ is a nonzero Lagrange multiplier. Now by partitioning $\tilde{Q}$ and $\tilde{P}$ as

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